Spin Stability of Torque-Free Systems—Part II

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Part I of this paper contains the development of a method for constructing stability criteria applicable to spinning motions of torque-free, elastic, dissipative systems possessing a finite number of degrees of freedom. In Part II, this method is applied to a spinning, rigid satellite that carries four elastically mounted antennas.

Introduction

THE concluding section of Part I contains a description of a procedure for the formulation of stability conditions. This procedure consists of six steps, designated (1–6), the first five of which suffice for the purpose of establishing stability. These five steps, numbered as such, are carried out in the sequel.

Analysis

1) Figure 1 shows a system S consisting of a "main body," B, and four "antennas" A_r ($r = 1,\ldots,4$). B is a rigid body having three (in general) unequal central principal moments of inertia. A_r ($r = 1,\ldots,4$) are "slender" rigid bodies. That is, A_r possesses a "central axis," L_r, such that the mass center, $A_r{}^*$, of A_r lies on L_r; the moments of inertia of A_r about all lines passing through $A_r{}^*$ and normal to L_r are equal to each other; and the moment of inertia of A_r about L_r is negligible.

A_r ($r = 1,\ldots,4$) are attached to B at points P_r ($r = 1,\ldots,4$) whose coordinates relative to X_1, X_2, X_3, the principal axes of inertia of B for the mass center B^* of B, are respectively $(R_1, 0, R_3)$, $(R_1, R_2, 0)$, $(R_1, 0, -R_3)$, $(R_1, -R_2, 0)$. The connection between A_r and B is effected by means of a revolute joint, a linear torsional spring, and a linear, viscous, torsional damper. The orientations of the axes of the joints, the spring constants, and the damping constants associated with these connections are given in Table 1, as are the distance from P_r to $A_r{}^*$, the mass of A_r, and the moment of inertia of A_r about any line passing through $A_r{}^*$ and normal to L_r. Body B has a mass m and a moment of inertia B_i about X_i ($i = 1, 2, 3$).

The system has four internal degrees of freedom. As we shall be concerned with motions during which the angles between L_r ($r = 1,\ldots,4$) and X_1 remain constant, we use as generalized coordinates, q_r ($r = 1,\ldots,4$), the radian measures of the deviations of the angles between L_r and X_1 during a general motion from the constants values of interest, denoting the latter by α in the case of A_1 and A_3, and by β for A_2 and A_4, as indicated in Fig. 1. If the springs at P_r ($r = 1,\ldots,4$) are undeformed when $\alpha+q_r$ ($r = 1, 3$) and $\beta+q_r$ ($r = 2, 4$) are equal to zero, the system of forces transmitted by B to A_r is then equivalent to a force whose line of action passes through P_r, together with a couple whose torque, $\mathbf{T}_r$ ($r = 1,\ldots,4$), is given by

$$\mathbf{T}_1 = -[k_1(\alpha+q_1)+\mu_1\dot{q}_1]\mathbf{b}_2 \tag{1a}$$

$$\mathbf{T}_2 = [k_2(\beta+q_2)+\mu_2\dot{q}_2]\mathbf{b}_3 \tag{1b}$$

Table 1 System parameters

	$r=1$	$r=2$	$r=3$	$r=4$
Orientation of joint axis at P_r	X_2	X_3	X_2	X_3
Spring constant at P_r	k_1	k_2	k_1	k_2
Damping constant at P_r	μ_1	μ_2	μ_1	μ_2
Distance from P_r to $A_r{}^*$	a	b	a	b
Mass of A_r	M	N	M	N
Transverse moment of inertia of A_r	I	J	I	J

$$\mathbf{T}_3 = [k_1(\alpha+q_3)+\mu_1\dot{q}_3]\mathbf{b}_2 \tag{1c}$$

$$\mathbf{T}_4 = -[k_2(\beta+q_4)+\mu_2\dot{q}_4]\mathbf{b}_3 \tag{1d}$$

where $\mathbf{b}_i$ ($i = 1,2,3$) are unit vectors directed as shown in Fig. 1. The corresponding generalized forces are

$$F_{q_r} = -[k_1(\alpha+q_r)+\mu_1\dot{q}_r] \quad r = 1,3 \tag{2a}$$

$$F_{q_r} = -[k_2(\beta+q_r)+\mu_2\dot{q}_r] \quad r = 2,4 \tag{2b}$$

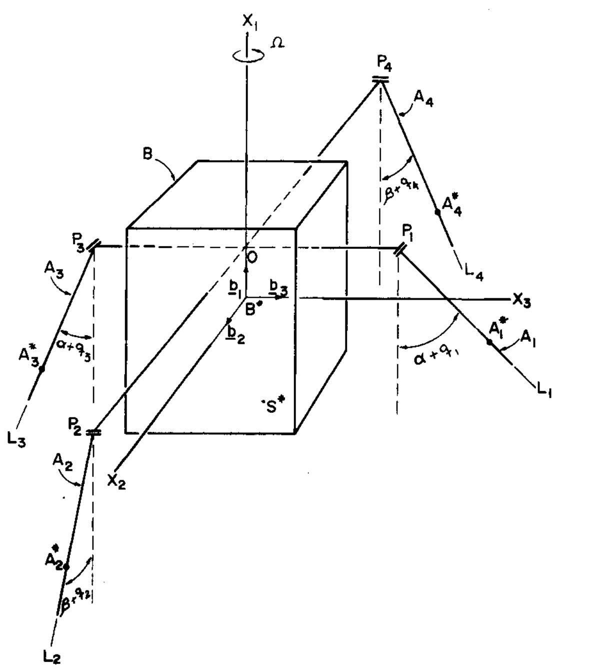

Fig. 1 Four antennae system.

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The requirements imposed by Eqs. [(2–4) Part I] can, therefore, be satisfied by defining V and D_r as

$$V \triangleq \tfrac{1}{2}k_1[(\alpha+q_1)^2+(\alpha+q_3)^2]+\tfrac{1}{2}k_2[(\beta+q_2)^2+(\beta+q_4)^2] \quad (3)$$

and

$$D_r \triangleq -\mu_1\dot{q}_r \quad r=1,3. \qquad D_r \triangleq -\mu_2\dot{q}_r \quad r=2,4 \quad (4)$$

2) To describe a simple spin which proceeds with a constant angular speed Ω and throughout which the angular momentum of S relative to the mass center, S^*, of S has a constant magnitude H, we let $\boldsymbol{\omega}$ denote the angular velocity of B in a newtonian reference frame and, after defining ω_i $(i=1,2,3)$ as

$$\omega_i \triangleq \boldsymbol{\omega}\cdot\mathbf{b}_i \quad i=1,2,3 \quad (5)$$

take

$$\omega_1=\Omega,\ \omega_2=\omega_3=0,\ q_r=0 \qquad r=1,\ldots,4 \quad (6)$$

For Z_1, Z_2, Z_3, we take axes originating at S^* and respectively parallel to X_1, X_2, X_3, since the latter axes are principal axes of S for S^* when $q_r=0$ $(r=1,\ldots,4)$. Next, direction cosine matrices and position vector components suitable for use in Eqs. [(79–81) Part I] can be formulated for each body, in which connection it is helpful to introduce the following abbreviations:

$$s_r \triangleq \begin{cases}\sin(\alpha+q_r) & r=1,3\\ \sin(\beta+q_r) & r=2,4\end{cases} \quad (7a)$$

$$c_r \triangleq \begin{cases}\cos(\alpha+q_r) & r=1,3\\ \cos(\beta+q_r) & r=1,4\end{cases} \quad (7b)$$

$$Q \triangleq m+2(M+N) \quad (8)$$

For B

$$[C_{jk}{}^{B}] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad (9a)$$

$$p_1{}^B = -R_1+Q^{-1}[mR_1+aM(c_1+c_3)+bN(c_2+c_4)] \quad (9b)$$

$$p_2{}^B = -aMQ^{-1}(s_1-s_3),\ p_3{}^B = -bNQ^{-1}(s_2-s_4) \quad (9c)$$

while for A_1

$$[C_{jk}{}^{A_1}] = \begin{bmatrix}c_1&0&s_1\\0&1&0\\-s_1&0&c_1\end{bmatrix} \quad (10a)$$

$$p_1{}^{A_1} = -bc_1+Q^{-1}[mR_1+aM(c_1+c_3)+bN(c_2+c_4)] \quad (10b)$$

$$p_2{}^{A_1} = -aMQ^{-1}(s_1-s_3),\ p_3{}^{A_1} = -bNQ^{-1}(s_2-s_4)+R_3+as_1 \quad (10c)$$

and similarly for $A_2,\ldots,A_4$. Using Eq. (79) (Part I), and letting

$$s\alpha \triangleq \sin\alpha,\ c\alpha \triangleq \cos\alpha,\ s\beta \triangleq \sin\beta,\ c\beta \triangleq \cos\beta \quad (11)$$

one thus arrives at

$$\tilde{I}_{11} = B_1+2[Is^2\alpha+Js^2\beta+M(R_3+as\alpha)^2+N(R_2+bs\beta)^2] \quad (12a)$$

$$\tilde{I}_{22} = B_2+2\{I+Jc^2\beta+N[R_1mQ^{-1}-b(1-2NQ^{-1})c\beta+2aMQ^{-1}c\alpha]^2+M([R_1mQ^{-1}-a(1-2MQ^{-1})c\alpha+2bNQ^{-1}c\beta]^2+(R_3+as\alpha)^2)\} \quad (12b)$$

$$\tilde{I}_{33} = B_3+2\{J+Ic^2\alpha+M[R_1mQ^{-1}-a(1-2MQ^{-1})c\alpha+2bNQ^{-1}c\beta]^2+N([R_1mQ^{-1}-b(1-2NQ^{-1})c\beta+2aMQ^{-1}c\alpha]^2+(R_2+bs\beta)^2)\} \quad (12c)$$

while Eqs. (80) (Part I) furnish

$$\tilde{I}_{11,1} = \tilde{I}_{11,3} = 2[(I+a^2M)s\alpha+aR_3Mc\alpha] \quad (13a)$$

$$\tilde{I}_{11,2} = \tilde{I}_{11,4} = 2[(J+b^2N)s\beta+bR_2Nc\beta] \quad (13b)$$

$$\tilde{I}_{12,1} = \tilde{I}_{12,3} = \tilde{I}_{13,2} = \tilde{I}_{13,4} = 0 \quad (13c)$$

$$\tilde{I}_{12,2} = -\tilde{I}_{12,4} = Jc2\beta-N\{bs\beta(R_2+bs\beta)+R_1mQ^{-1}-b(1-2NQ^{-1})c\beta+2aMQ^{-1}c\alpha]b(1-2NQ^{-1})c\beta\}-2bNQ^{-2}[R_1mN+2aMNc\alpha-bN(2M+m)c\beta]c\alpha \quad (13d)$$

$$\tilde{I}_{13,1} = -\tilde{I}_{13,3} = Ic2\alpha-M\{as\alpha(R_3+as\alpha)+[R_1mQ^{-1}-a(1-2MQ^{-1})c\alpha+2bNQ^{-1}c\beta]a(1-2MQ^{-1})c\alpha\}-2aMQ^{-2}[R_1mM+2bMNc\beta-aM(2N+m)c\alpha]c\beta \quad (13e)$$

and Eqs. (81) (Part I) yield

$$\tilde{I}_{11,12} = \tilde{I}_{11,23} = \tilde{I}_{11,34} = \tilde{I}_{11,41} = 0 \quad (14a)$$

$$\tilde{I}_{11,11} = \tilde{I}_{11,33} = 2[Ic2\alpha-aM(R_3+as\alpha)s\alpha+a^2M(1-MQ^{-1})c^2\alpha] \quad (14b)$$

$$\tilde{I}_{11,22} = \tilde{I}_{11,44} = {}^2[Jc2\beta-bN(R_2+bs\beta)s\beta+b^2N(1-NQ^{-1})c^2\beta] \quad (14c)$$

$$\tilde{I}_{11,33} = 2a^2M^2Q^{-1}c^2\alpha,\quad \tilde{I}_{11,24} = 2b^2N^2Q^{-1}c^2\beta \quad (14d)$$

3) To determine the circumstances under which the simple spin described by Eq. (6) can occur, we note that, from Eq. (3)

$$\tilde{V}_{,1} = \tilde{V}_{,3} = k_1\alpha,\quad \tilde{V}_{,2} = \tilde{V}_{,4} = k_2\beta \quad (15)$$

Using these results, Eqs. (13), and

$$H^2 = (\tilde{I}_{11}\Omega)^2 \quad (16)$$

in Eq. (84) (Part I) with $r=1$ or $r=3$, we then obtain

$$k_1 = \Omega^2\alpha^{-1}[Is\alpha c\alpha+aMc\alpha(R_3+as\alpha)] \quad (17a)$$

while for $r=2$ or $r=4$,

$$k_2 = \Omega^2\beta^{-1}[Js\beta c\beta+bNc\beta(R_2+bs\beta)] \quad (17b)$$

The simple spin under consideration thus can occur only when these equations are satisfied. In this connection it should be kept in mind that k_1, and k_2 are intrinsically positive.

We note for future reference that, from Eq. (3),

$$\tilde{V}_{,11} = \tilde{V}_{,33} = k_1,\quad \tilde{V}_{,22} = \tilde{V}_{,44} = k_2 \quad (18a)$$

$$\tilde{V}_{,rs} = 0 \quad \text{if} \quad r \neq s \quad (18b)$$

4) The *external stability conditions* [see the inequalities (82) (Part I)]

$$\tilde{I}_{11}-\tilde{I}_{22} > 0,\quad \tilde{I}_{11}-\tilde{I}_{33} > 0 \quad (19)$$

can now be expressed in terms of system parameters by substitution from Eq. (12).

5) All ingredients required for the evaluation of $\tilde{Z}_{,rs}$ [see Eq. (83) Part I] are available in Eqs. (12–18); and, if Δ_r $(r=1,\ldots,4)$ are defined as

$$\Delta_1 \triangleq \tilde{Z}_{,11},\quad \Delta_2 \triangleq \begin{vmatrix}\tilde{Z}_{,11} & \tilde{Z}_{,12}\\ \tilde{Z}_{,21} & \tilde{Z}_{,22}\end{vmatrix} \quad (20a)$$

$$\Delta_3 \triangleq \begin{vmatrix}\tilde{Z}_{,11} & \tilde{Z}_{,12} & \tilde{Z}_{,13}\\ \tilde{Z}_{,21} & \tilde{Z}_{,22} & \tilde{Z}_{,23}\\ \tilde{Z}_{,31} & \tilde{Z}_{,32} & \tilde{Z}_{,33}\end{vmatrix},\quad \Delta_4 \triangleq \begin{vmatrix}\tilde{Z}_{,11} & \tilde{Z}_{,12} & \tilde{Z}_{,13} & \tilde{Z}_{,14}\\ \tilde{Z}_{,21} & \tilde{Z}_{,22} & \tilde{Z}_{,23} & \tilde{Z}_{,24}\\ \tilde{Z}_{,31} & \tilde{Z}_{,32} & \tilde{Z}_{,33} & \tilde{Z}_{,34}\\ \tilde{Z}_{,41} & \tilde{Z}_{,42} & \tilde{Z}_{,43} & \tilde{Z}_{,44}\end{vmatrix} \quad (20b)$$

then, in accordance with Sylvester's criteria, the *internal stability conditions* [see the inequalities (63) (Part I)] can be stated as

$$\Delta_r > 0 \qquad (r=1,\ldots,4) \quad (21)$$

Hence, given the values of the "rest" angles α and β, the inertia parameters m, B_1, B_2, B_3 for body B, the hinge point location parameters R_1, R_2, R_3, and the inertia parameters a, b, M, N, I, J for the antennae A_r $(r=1,\ldots,4)$, one can now establish stability of the simple spin under consideration by means of elementary calculations. For example, suppose $\alpha=\beta=60°$; $m=500, B_1=110, B_2=100, B_3=70$; $R_1=0.5, R_2=R_3=0.3$; and $a=b=0.2, M=N=2, I=J=0.02$. Here, as in the sequel, all lengths, masses, and moments of inertia are expressed in units of meters, kilograms, and kilogram-(meters)^2, respectively. Then, for instance,

$$\underset{(12a)}{\tilde{I}_{11}=111.8},\quad \underset{(13b)}{\tilde{I}_{11,2}=0.207},\quad \underset{(14c)}{\tilde{I}_{11,22}=-0.308}$$

$$\underset{(17a)}{k_1=0.09864\Omega^2},\quad \underset{(15)}{\tilde{V}_{,2}=0.1033\Omega^2},\quad \underset{(18a)}{\tilde{V}_{,44}=0.0986\Omega^2}$$

$$\underset{(16)}{H^2=12511\cdot\Omega^2},\quad \underset{(9.2)}{\tilde{Z}_{,33}=0.2514\Omega^2}$$

From the inequalities (19), the external stability conditions are

$$9.646 > 0,\quad 39.65 > 0$$

and these are clearly satisfied. And from Eq. (20),

$$\Delta_1 = 0.2514\Omega^2,\quad \Delta_2 = 0.06195\Omega^4$$

$$\Delta_3 = 0.01557\Omega^6,\quad \Delta_4 = 0.003835\Omega^8$$

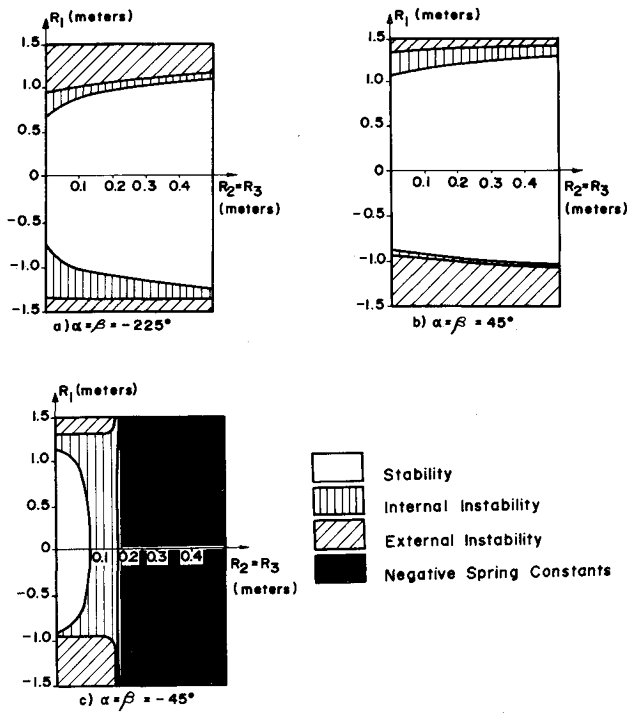

Fig. 2 Four antennae system stability charts.

Hence the internal stability conditions [the inequalities (21)] also are satisfied, and we are dealing with a stable simple spin.

The method at hand is particularly well suited for dealing with questions of the kind encountered in preliminary design studies. For example, one may wish to explore alternative schemes for the deployment of the antennae, given the inertia properties of both the main body and the antennae. Suppose, for instance, that with m, B_1, B_2, B_3, a, b, M, N, I, and J as before one wished to place the points P_r ($r = 1, \ldots, 4$) (see Fig. 1) in such a way that $\alpha = \beta = -225°$ be permissible. (This sort of configuration is attractive because it permits use of the torsion springs at P_r ($r = 1, \ldots, 4$) for *two* purposes: pressing the antennae against the main body during "spin up," that is, while Ω is made to increase from zero to its design value; and holding the antennae in position when Ω has attained the design value.) Figure 2a shows a stability chart obtained by using the inequalities (19) and (20) with the given parameter values for various values of R_1 and $R_2 = R_3$. Horizontal shading indicates that at least one of the inequalities (19) is violated; vertical shading corresponds to failure to satisfy all of the inequalities (21); and for each point of the unshaded region simple spin about X_1 is a stable motion. With this chart in hand, one can place P_r ($r = 1, \ldots, 4$) in an infinite number of permissible ways; and the value of Ω associated with a particular placement of P_r ($r = 1, \ldots, 4$) can be assigned at will, provided k_1 and k_2 be chosen in accordance with Eq. (17).

Questions regarding the relative merits of various antennae arrangements fall outside of the limits of the present work. However, it is worth mentioning in conclusion that the system under consideration offers a considerable variety of possible arrangements. This is apparent from Fig. 2, which indicates that there exist stable spins about X_1 in the following ranges: $-270° < \alpha = \beta \leqq -180°$ (Fig. 2a), $0° < \alpha = \beta \leqq 90°$ (Fig. 2b), and $-90° \leqq \alpha = \beta < 0°$ (Fig. 2c). (For the parameter values used to construct these stability charts, and in the given ranges for R_1 and $R_2 = R_3$, we have found no stable spins about X_1 for $-90° < \alpha = \beta < 0°$. The solidly black portion in Fig. 2c corresponds to configurations that must be ruled out because the associated values of k_1 and k_2 are negative. In any event, this chart deals with a situation that is of more theoretical than practical interest.) Finally, we should like to stress that it is not necessary to take $\alpha = \beta$, $R_2 = R_3$, $a = b$, $M = N$, and $I = J$. This was done in connection with Fig. 2 solely for the purpose of minimizing the complexity of an illustrative example.